

MEAN-FIELD STUDY OF  $^{12}\text{C}+^{12}\text{C}$  FUSION

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**Abstract.** *The nuclear mean-field potential arising from the  $^{12}\text{C}+^{12}\text{C}$  interaction at the low energies relevant for the astrophysical carbon burning process has been constructed within the double-folding model, using the realistic nuclear ground-state density of the  $^{12}\text{C}$  nucleus and the effective M3Y nucleon-nucleon (NN) interaction constructed from the G-matrix of the Paris (free) NN potential. To explore the nuclear medium effect, both the original density independent M3Y-Paris interaction and its density dependent CDM3Y6 version have been used in the folding model calculation of the  $^{12}\text{C}+^{12}\text{C}$  potential. The folded potentials at the different energies were used in the optical model description of the elastic  $^{12}\text{C}+^{12}\text{C}$  scattering at the energies around and below the Coulomb barrier, as well as in the barrier penetration model to estimate the fusion cross section and astrophysical S factor of the  $^{12}\text{C}+^{12}\text{C}$  reactions at the low energies. The obtained results are in good agreement with experimental data over a wide range of energies.*

**Key words.** *Nuclear mean-field,  $^{12}\text{C}+^{12}\text{C}$  fusion, astrophysical S factor.*

## I. INTRODUCTION

The  $^{12}\text{C}+^{12}\text{C}$  fusion plays an important role in the whole chain of nucleosynthesis processes during stellar evolution, as the main nuclear reaction governing the carbon burning process in the young massive stars that generates the heavier elements or the pycnonuclear reaction that leads a carbon-oxygen white dwarf to the type Ia supernova explosion [1, 2, 3]. A general scenario for a massive star of about ten times the solar mass implies that after the helium burning process its core consists predominantly of the  $^{12}\text{C}$  and  $^{16}\text{O}$  ashes. As soon as this core begins to collapse gravitationally igniting the  $^{12}\text{C}$  and  $^{16}\text{O}$  ashes into the  $^{12}\text{C}+^{12}\text{C}$ ,  $^{12}\text{C}+^{16}\text{O}$ , and  $^{16}\text{O}+^{16}\text{O}$  fusion reactions, the first reaction is more favorable because it has the lowest Coulomb barrier. It also generates the heavier nuclei like  $^{23}\text{Na}$ ,  $^{20}\text{Ne}$ , and  $^{23}\text{Mg}$  for the next burning stage of the stellar evolution. In fact, at the typical temperatures and densities in the outer-shell region of  $T \approx 10^9$  K and  $\rho \approx 10^5$  g/cm<sup>3</sup> respectively, the  $^{12}\text{C}+^{12}\text{C}$  fusion forms the  $^{24}\text{Mg}$  compound nucleus with mass difference between the  $^{12}\text{C}+^{12}\text{C}$  system and  $^{24}\text{Mg}$  nucleus of about 14 MeV. Therefore, the compound  $^{24}\text{Mg}^*$  nucleus is highly excited and has a large number of overlapping states with the partial widths of the light particle emissions (neutron, proton and  $\alpha$ ) dominating the  $\gamma$ -ray emission width. The main decay products of the compound  $^{24}\text{Mg}^*$  nucleus are  $^{23}\text{Na}$ ,  $^{20}\text{Ne}$ , and  $^{23}\text{Mg}$  in the  $^{12}\text{C}(^{12}\text{C},p)^{23}\text{Na}$  ( $Q = 2241$  keV),  $^{12}\text{C}(^{12}\text{C},\alpha)^{20}\text{Ne}$  ( $Q = 4617$  keV), and  $^{12}\text{C}(^{12}\text{C},n)^{23}\text{Mg}$  ( $Q = -2599$  keV) reaction channels, respectively,

while the remaining processes such as  $^{12}\text{C}(^{12}\text{C},\gamma)^{24}\text{Mg}$ ,  $^{12}\text{C}(^{12}\text{C},^8\text{Be})^{16}\text{O}$  are less important at astrophysical energies [3]. In these channels, the  $^{12}\text{C}(^{12}\text{C},p)^{23}\text{Na}$  and  $^{12}\text{C}(^{12}\text{C},\alpha)^{20}\text{Ne}$  reactions dominate the total  $^{12}\text{C}+^{12}\text{C}$  fusion cross section with about equal probabilities for proton and  $\alpha$  emissions.

In astrophysical conditions, the effective thermal energy of  $^{12}\text{C}$  is approximately 2 MeV [1] while the Coulomb barrier of the  $^{12}\text{C}+^{12}\text{C}$  system is around 8 MeV which substantially lowers the probability of  $^{12}\text{C}+^{12}\text{C}$  fusion in such conditions. A narrow window for  $^{12}\text{C}+^{12}\text{C}$  fusion becomes possible thanks to the quantum tunneling effect that allows the two  $^{12}\text{C}$  nuclei to penetrate the Coulomb barrier without the need of having sufficient energy to overcome it [4]. The  $^{12}\text{C}+^{12}\text{C}$  fusion caused by the tunnel effect has been reasonably described by the barrier penetration model (BPM) [5, 6, 7], which is used by many authors to estimate nuclear reaction rates in stars. Typically, the nuclear reaction rate, a vital input for the study of stellar evolution, is expressed in terms of the astrophysical  $S$  factor [8]

$$S = E_{\text{c.m.}} \sigma_{\text{R}} \exp(2\pi\eta), \quad (1)$$

here  $E_{\text{c.m.}}$  is the center-of-mass (c.m.) kinetic energy (in MeV) in the entrance channel,  $\sigma_{\text{R}}$  is the total reaction cross section (in barn) and  $\eta$  is the Sommerfeld parameter determined as

$$2\pi\eta = 2\pi \frac{Z_1 Z_2 e^2}{\hbar\nu} = \frac{87.2}{\sqrt{E_{\text{c.m.}}(\text{MeV})}}, \quad (2)$$

where  $\nu$  is the relative velocity of the colliding nuclei. The astrophysical  $S$  factor is an important quantity introduced to describe the rate of a specific reaction in nuclear astrophysics studies [9]. At very low energies, typical for nuclear astrophysics processes, the cross sections (or the astrophysical  $S$  factors) of the charged-induced reactions are extremely difficult to measure in the laboratory because of the repulsive Coulomb barrier that reduces the  $S$  factor substantially. Therefore, it is important to have a reliable theoretical model to evaluate the astrophysical  $S$  factors of different nuclear reactions in the stellar energy region.

Because the  $^{12}\text{C}+^{12}\text{C}$  fusion reaction is an important part of the star evolution and still not fully understood, it has motivated many studies during the last four decades [10, 11, 12, 13, 14, 15, 16, 17]. The cross section of the  $^{12}\text{C}+^{12}\text{C}$  fusion reaction was calculated by different authors within the BPM framework using the different models of the  $^{12}\text{C}+^{12}\text{C}$  potential [18, 19, 20, 21, 22]. However, the physics origin of the rapidly fluctuating  $^{12}\text{C}+^{12}\text{C}$  fusion cross section observed at the lowest energies remains unexplained and needs to be further investigated.

In general, the nucleus-nucleus potential in the low-energy region can be naturally associated with the nuclear mean field formed during the dinuclear collision [23]. As a result, the so-called double folding model (DFM) which evaluates the nucleus-nucleus potential as the Hartree-Fock potential uses a realistic effective nucleon-nucleon (NN) interaction and the nuclear density distributions of the two colliding nuclei [24, 25]. In the present paper, we explore the applicability of the DFM to determine the nuclear mean-field potential of the  $^{12}\text{C}+^{12}\text{C}$  system in the very low energy range (2-10 MeV), typical of  $^{12}\text{C}+^{12}\text{C}$  fusion, using both the original M3Y-Paris interaction [26] (constructed to

reproduce the G-matrix elements of the Paris NN potential [27] in an oscillator basis) and its CDM3Y6 density dependent version [28]. The  $^{12}\text{C}+^{12}\text{C}$  potential obtained in the DFM is further used in the BPM to calculate the cross section and astrophysical  $S$  factor of the  $^{12}\text{C}+^{12}\text{C}$  fusion reaction.

The paper is organized as follows. In the next section, we give a brief introduction to the theoretical methods used in this paper. The numerical results and discussions are given in the Sec. III. The last section summarizes the main results of the present work.

## II. THEORETICAL METHODS

### II.1. The WKB method in the barrier penetration model

The Wentzel-Kramers-Brillouin (WKB) method is well known to provide a semi-classical approximation for the solution of the Schrödinger equation. As such, the WKB method has been used to elaborate the physics treatment of the BPM for nucleus-nucleus interacting systems at very low energies, where the nuclear mean-field potentials vary slowly over a spatial region of the order of the system wavelength [5, 19, 20]. In particular, the  $^{12}\text{C}+^{12}\text{C}$  fusion reaction can be studied within the BPM based on the simple treatment of the WKB method.

In general, the nucleus-nucleus interaction potential consists of the nuclear, centrifugal, and Coulomb terms

$$V(r) = V_N(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} + V_C(r), \quad (3)$$

where  $l$  is the orbital angular momentum and  $\mu = \frac{mA}{2}$  is the reduced mass of the nucleus-nucleus system, and  $m$  is the free nucleon mass. The Coulomb potential  $V_C$  is usually assumed [29] as

$$V_C(r) = \begin{cases} \frac{Z^2 e^2}{r} & \text{if } r > R_c \\ \left(3 - \frac{r^2}{R_c^2}\right) \frac{Z^2 e^2}{2R_c} & \text{if } r \leq R_c \end{cases} \quad (4)$$

where  $R_c = 2r_c A^{1/3}$  with  $A$  being the mass number, and  $r_c = 0.95$  fm. The  $l$ -dependent centrifugal potential is that arising in the Schrödinger equation with spherically symmetric central potential. The nuclear potential  $V_N$  given by the DFM calculation is used in the present work to determine the total nucleus-nucleus potential.

Within the BPM [6], the fusion cross section of the particle flux transmitted through the Coulomb barrier is obtained from the  $l$ -dependent transmission coefficients  $T_l$  as

$$\sigma_R = \frac{\pi}{k^2} \sum_{l=0}^{l_{\text{cr}}} (2l+1) T_l, \quad (5)$$

where  $k$  is the relative momentum,  $l_{\text{cr}}$  is the critical angular momentum corresponding to the largest value of the orbital angular momentum that reproduces both the pocket and barrier of the total nucleus-nucleus potential (3).  $V_{Bl}$  is the barrier height, i.e., the value

of the total nucleus-nucleus potential at the barrier radius  $V(r = R_{Bl})$ , which is different from the Coulomb barrier.

For the partial waves  $l$  with  $V_{Bl} < E_{c.m.}$ , the shape of the nucleus-nucleus potential around  $R_{Bl}$  can be approximated as a parabola with the curvature determined as

$$\hbar\omega_l = \left| \frac{\hbar^2}{\mu} \frac{d^2 V}{dr^2} \right|_{R_{Bl}}^{1/2}. \quad (6)$$

Then, the transmission coefficient  $T_l$  is obtained from the Hill-Wheeler formula [30] as

$$T_l = \left[ 1 + \exp \left( \frac{2\pi[V_{Bl} - E_{c.m.}]}{\hbar\omega_l} \right) \right]^{-1}. \quad (7)$$

For the partial waves  $l$  with  $V_{Bl} > E_{c.m.}$ ,  $T_l$  is determined based on the WKB approximation

$$T_l = [1 + \exp(S_l)]^{-1}, \quad (8)$$

$$S_l = \int_{r_1}^{r_2} \sqrt{\frac{8\mu}{\hbar^2} [V(r) - E_{c.m.}]} dr, \quad (9)$$

here  $r_1, r_2$  are the radii of the classical turning points where  $V(r_{1(2)}) = E_{c.m.}$ .

## II.2. Double-folding model of the nucleus-nucleus potential

In the framework of the BPM and the nuclear optical model [31], the nuclear part of the total nucleus-nucleus potential is the most important input. From the physics point of view, it is always of interest to determine  $V_N$  starting from the nucleon degrees of freedom, and the double-folding model [24, 25] is the most commonly used approach for that purpose. In this model,  $V_N$  is evaluated as the Hartree-Fock (HF) potential with an appropriately chosen effective NN interaction between nucleons in the target and projectile

$$V_N = V_D + V_{EX} = \sum_{i \in A_1, j \in A_2} [\langle ij | v_D | ij \rangle + \langle ij | v_{EX} | ji \rangle], \quad (10)$$

Treating explicitly the single-nucleon wave functions in the HF potential (10), the local direct term is reduced to a double-folding integration of the densities of the two colliding nuclei with the direct part of the NN interaction

$$V_D(r) = \int \rho_1(\mathbf{r}_1) \rho_2(\mathbf{r}_2) v_D(\rho, s) d^3 r_1 d^3 r_2, \quad \mathbf{s} = \mathbf{r}_2 - \mathbf{r}_1 + \mathbf{r}. \quad (11)$$

The antisymmetrization gives rise to the exchange term in Eq. (10) which is, in general, nonlocal. An accurate local equivalent exchange potential can be obtained [25] using the local WKB approximation [29] for the change in relative motion induced by the exchange of the spatial coordinates of each interacting nucleon pair

$$V_{EX}(r) = \int \rho_1(\mathbf{r}_1, \mathbf{r}_1 + \mathbf{s}) \rho_2(\mathbf{r}_2, \mathbf{r}_2 - \mathbf{s}) v_{EX}(\rho, s) \exp \left( \frac{i\mathbf{k}(r)\mathbf{s}}{M} \right) d^3 r_1 d^3 r_2. \quad (12)$$

Here  $\mathbf{k}(r)$  is determined as

$$k^2(r) = \frac{2\mu}{\hbar^2} [E_{c.m.} - V_N(r) - V_C(r)], \quad (13)$$

where  $M = 2A$ ,  $V_N(r)$  and  $V_C(r)$  are the nuclear and Coulomb parts of the total nucleus-nucleus potential, respectively, and  $\rho_{1(2)}(\mathbf{r}, \mathbf{r}')$  is the single-nucleon density matrix. It can be seen from Eqs. (10)-(13) that the DFM calculation of the nucleus-nucleus potential (10) is a self-consistent problem. Therefore, the calculation of  $V_{\text{EX}}$  is carried out iteratively based on a realistic expansion method for the density matrix [25].

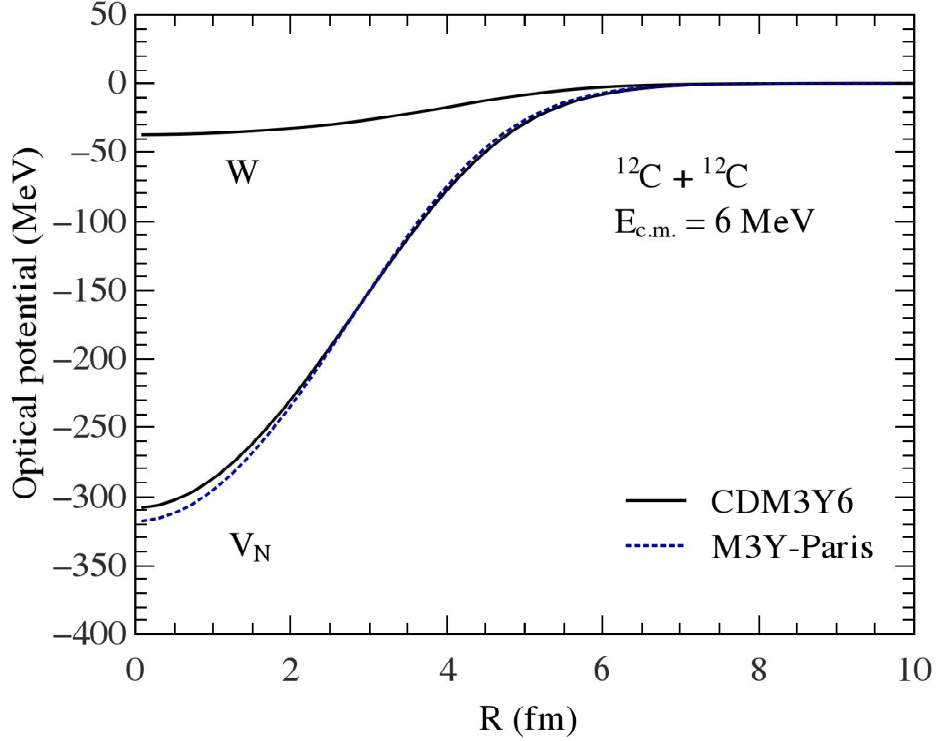
Among different choices of the effective NN interaction, a density dependent version of the M3Y-Paris interaction (dubbed as CDM3Y6 interaction [28]) has been used quite successfully in the folding model analyses of elastic and inelastic nucleus-nucleus scattering. The density dependent parameters of the CDM3Y6 interaction were carefully adjusted in the HF scheme to reproduce the saturation properties of nuclear matter [28]. In the present work, both the CDM3Y6 and original density independent M3Y-Paris interactions were used in the DFM calculation. To avoid a phenomenological choice of the imaginary part of the nuclear optical potential, the CDM3Y6 interaction has been supplemented with a realistic imaginary density dependence for the folding calculation of the imaginary potential. The parameters of the imaginary density dependence have been deduced at each energy based on the Brueckner Hartree-Fock results for the nucleon optical potential in nuclear matter by Jeukenne, Lejeune and Mahaux, widely known as the JLM potential [32]. Given an accurate choice of the effective NN interaction, the DFM can be applied successfully to calculate the nucleus-nucleus potential only if the realistic nuclear densities were used in the folding calculation (11)-(12). In the present work, the two-parameter Fermi function was used for the ground-state density of the  $^{12}\text{C}$  nucleus

$$\rho(r) = \rho_0 / \{1 + \exp[(r - c)/a]\}. \quad (14)$$

The parameters in Eq. (14) were chosen to reproduce reasonably the empirical nuclear root-mean-square radius based on elastic electron scattering data as well as the radial shape of the nuclear density given by the shell model calculations [24, 25]. Given the appropriate choice of the ground-state density of  $^{12}\text{C}$  and realistic density dependent NN interaction, the folded  $^{12}\text{C}+^{12}\text{C}$  potential (10)-(12) represents the mean-field potential [23] in the nuclear medium formed in the  $^{12}\text{C}+^{12}\text{C}$  collision. As such, the folded  $^{12}\text{C}+^{12}\text{C}$  potential can be used as the nuclear optical potential to study the elastic  $^{12}\text{C}+^{12}\text{C}$  scattering and to estimate the reaction rate of the  $^{12}\text{C}+^{12}\text{C}$  fusion in the BPM.

### III. RESULTS AND DISCUSSIONS

The reliability of the folded  $^{12}\text{C}+^{12}\text{C}$  potential should be tested first in the optical model description of elastic  $^{12}\text{C}+^{12}\text{C}$  scattering at low energies before using it in the BPM to determine the astrophysical factor  $S$  of  $^{12}\text{C}+^{12}\text{C}$  fusion. In the present work we have analyzed the elastic  $^{12}\text{C}+^{12}\text{C}$  scattering data measured at energies  $E_{\text{c.m.}} = 6-10$  MeV [14], using the complex optical potential given by the DFM calculation (10)-(12). The radial shapes of the real ( $V_N$ ) and imaginary ( $W$ ) potentials folded with the density dependent CDM3Y6 interaction, as shown in Fig. 1, are compared with the real potential obtained with the density independent M3Y-Paris interaction. One can see that the medium effects given by the density dependence of the NN interaction make the real optical potential slightly shallower in the center but more attractive at the potential surface.



**Fig. 1.** The complex  $^{12}\text{C}+^{12}\text{C}$  optical potential folded with the density dependent CDM3Y6 interaction (solid line) at  $E_{\text{c.m.}} = 6$  MeV is compared with the real potential folded with the density independent M3Y-Paris interaction (dashed line).

The (energy-dependent) complex folded CDM3Y6 potential can be used as the optical potential to study elastic  $^{12}\text{C}+^{12}\text{C}$  scattering at low energies, relevant for nuclear astrophysics. In the present work, we have considered five elastic scattering angular distributions measured in  $^{12}\text{C}+^{12}\text{C}$  collisions at energies around the Coulomb barrier [14]. To fine tune the complex strength of the optical potential, a slight renormalization is usually adopted for the best optical model fit of the experimental data. Thus, the complex optical potential used as input for the Schrödinger equation has the form

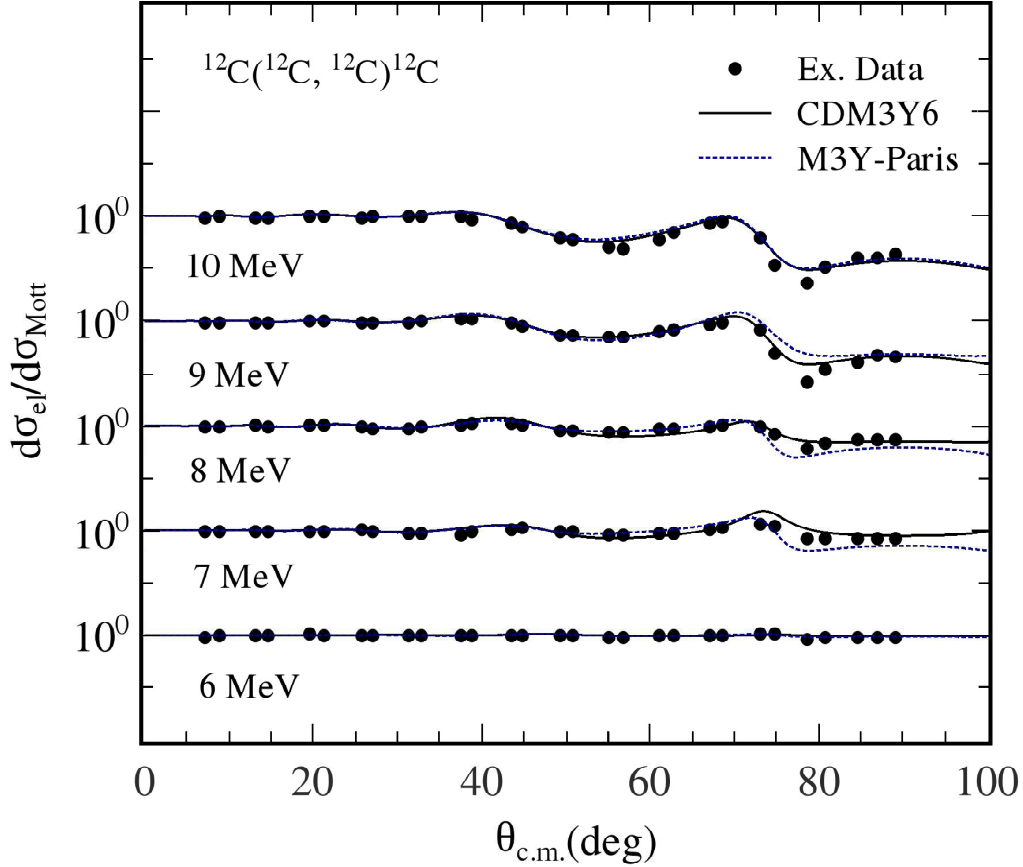
$$U(r, E) = N_r V_N(r, E) + i N_i W(r, E). \quad (15)$$

Very good optical model description of the considered elastic  $^{12}\text{C}+^{12}\text{C}$  data has been obtained with the complex folded CDM3Y6 potential renormalized by  $N_r \approx 0.85$  and  $N_i \approx 1.0$  (see Fig. 2).

One can see from the results plotted in Fig. 2 that the complex folded CDM3Y6 potential gives a very good description of the elastic  $^{12}\text{C}+^{12}\text{C}$  data at low energies. The effect on the real optical potential caused by the density dependence of the CDM3Y6 interaction shows up in the difference between the results given by the M3Y-Paris interaction (dashed line in Fig. 2) and those given by the CDM3Y6 interaction (solid line). Note that these optical model calculations used the same imaginary part of the optical potential

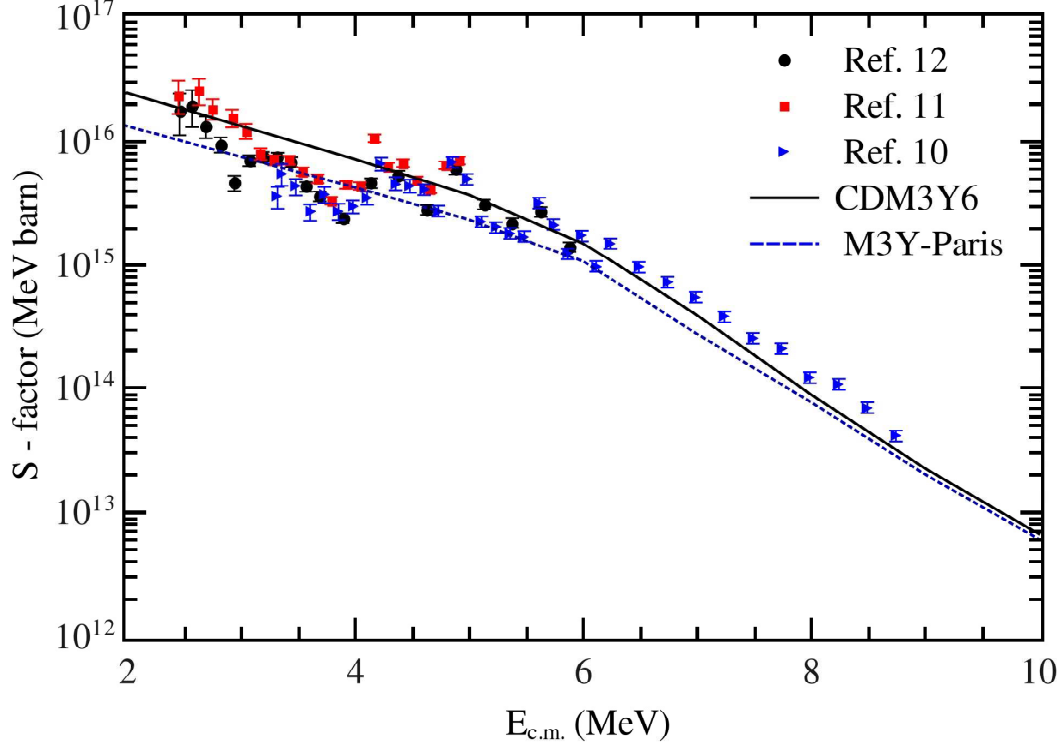
obtained with the complex density dependent CDM3Y6 interaction. We found that the inclusion of the realistic density dependence into the effective NN interaction discussed widely in Refs. [25, 28] is also necessary for the good optical model description of elastic  $^{12}\text{C} + ^{12}\text{C}$  scattering at low energies. It remains now to be seen whether this effect can also be observed in the BPM calculation of the  $^{12}\text{C} + ^{12}\text{C}$  fusion reactions at very low energies.

The nuclear folded M3Y-Paris and CDM3Y6 potentials have been further used in the BPM to evaluate the  $^{12}\text{C} + ^{12}\text{C}$  fusion cross section using Eqs. (3)-(9), and the results were used in Eq. (1) to calculate the astrophysical  $S$  factor of the  $^{12}\text{C} + ^{12}\text{C}$  fusion reaction. The results obtained for the  $S$  factor are shown in Fig. 3 and one can see a reasonably good agreement of the BPM results with the experimental data [10, 11, 12] over a wide range of energies. However, the slight wiggling behavior of the experimental  $S$  factor in the energy



**Fig. 2.** The elastic  $^{12}\text{C} + ^{12}\text{C}$  scattering cross sections (as ratio to the Mott cross section of the Coulomb scattering of the two identical charged particles) at energies around the Coulomb barrier. The dashed and solid curves are the results of the optical model calculation using the M3Y-Paris and CDM3Y6 interactions, respectively. The experimental data are taken from Ref. [14].





**Fig. 3.** The astrophysical  $S$  factor as a function of the center-of-mass energy. The dashed and solid curves represent the BPM results given by the M3Y-Paris and CDM3Y6 interactions, respectively. The experimental data are taken from Refs. [10, 11, 12].

range of 2 to 5 MeV (see Fig. 3) cannot be reproduced by the BPM using the mean-field potential of the  $^{12}\text{C}+^{12}\text{C}$  system. Such an oscillation of the  $S$  factor in this energy range has been discussed as a resonant behavior of the  $^{12}\text{C}+^{12}\text{C}$  fusion reaction [18, 33] caused by the relatively large spacings and narrow widths of the  $^{24}\text{Mg}^*$  compound levels [34]. Although our mean-field approach to the  $^{12}\text{C}+^{12}\text{C}$  potential does not include any resonance effect, the average description of the  $S$  factor by the folded CDM3Y6 potential is quite satisfactory, so that the mean-field prediction of the  $^{12}\text{C}+^{12}\text{C}$  reaction rate is accurate over the entire Gamow range.

The results of the BPM calculation shown in Fig. 3 also show that the  $S$  factors obtained with the density independent M3Y-Paris interaction are somewhat lower than the experimental data and those obtained with the density dependent CDM3Y6 interaction. Technically it is explained by the fact that the M3Y-Paris potential provides a higher barrier in comparison with that given by the CDM3Y6 potential. Thus, the medium effects caused by the density dependence of the effective NN interaction are not negligible



in the BPM calculation of the  $S$  factor, and this conclusion is natural in view of the carbon burning process occurring in the dense baryon matter of very massive stars.

#### IV. SUMMARY

The nuclear mean-field potential arising in the  $^{12}\text{C}+^{12}\text{C}$  collision at very low energies has been constructed in the double-folding model using the realistic nuclear density and the complex density dependent CDM3Y6 interaction, based on the original M3Y-Paris interaction. The complex folded  $^{12}\text{C}+^{12}\text{C}$  potential was used in the optical model to successfully describe the elastic  $^{12}\text{C}+^{12}\text{C}$  scattering at the low energies around the Coulomb barrier. This same potential was shown to give also a realistic description of the astrophysical  $S$  factor for the  $^{12}\text{C}+^{12}\text{C}$  fusion reaction over a wide range of the energies.

The mean-field description of both the elastic scattering angular distributions and the  $S$  factor of the  $^{12}\text{C}+^{12}\text{C}$  reaction at the low energies has shown a rather strong medium effect caused by the density dependence of the effective NN interaction. The results obtained in the present work also confirmed the reliability of the double-folding model in the calculation of the total nuclear potential for the study of the  $^{12}\text{C}+^{12}\text{C}$  fusion reaction in the low-energy region of the nuclear astrophysical interest.

The further use of the DFM in the calculation of both the optical potential and inelastic scattering form factor is planned to be used within the framework of the coupled channel formalism for the study of inelastic scattering with the final state of  $^{12}\text{C}$  nucleus in excited states, such as  $2^+$  state at 4.44 MeV,  $0_2^+$  state at 7.65 MeV, and  $3^-$  state at 9.64 MeV. Besides, at the low energies around the Coulomb barrier the dominant final states of the  $^{12}\text{C}+^{12}\text{C}$  reaction are  $^{20}\text{Ne} + \alpha$ ,  $^{23}\text{Na} + p$  and  $^{16}\text{O} + ^8\text{Be}$ , and it is of high interest to estimate their explicit contributions to the total  $^{12}\text{C}+^{12}\text{C}$  reaction cross section in this energy range.

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